

EXTENSION THEORY OF SEPARABLE METRIZABLE SPACES WITH APPLICATIONS TO DIMENSION THEORY

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ABSTRACT. The paper deals with generalizing several theorems of the covering dimension theory to the extension theory of separable metrizable spaces. Here are some of the main results:

Generalized Eilenberg-Borsuk Theorem. *Let L be a countable CW complex. If X is a separable metrizable space and $K * L$ is an absolute extensor of X for some CW complex K , then for any map $f : A \rightarrow K$, A closed in X , there is an extension $f' : U \rightarrow K$ of f over an open set U such that $L \in AE(X - U)$.*

Theorem. *Let K, L be countable CW complexes. If X is a separable metrizable space and $K * L$ is an absolute extensor of X , then there is a subset Y of X such that $K \in AE(Y)$ and $L \in AE(X - Y)$.*

Theorem. *Suppose G_1, \dots, G_n are countable, non-trivial, abelian groups and $k > 0$. For any separable metrizable space X of finite dimension $\dim X > 0$, there is a closed subset Y of X with $\dim_{G_i} Y = \max(\dim_{G_i} X - k, 1)$ for $i = 1, \dots, n$.*

Theorem. *Suppose W is a separable metrizable space of finite dimension and Y is a compactum of finite dimension. Then, for any k , $0 < k < \dim W - \dim Y$, there is a closed subset T of W such that $\dim T = \dim W - k$ and $\dim(T \times Y) = \dim(W \times Y) - k$.*

Theorem. *Suppose X is a metrizable space of finite dimension and Y is a compactum of finite dimension. If $K \in AE(X)$ and $L \in AE(Y)$ are connected CW complexes, then $K \wedge L \in AE(X \times Y)$.*

1. INTRODUCTION

Recall that the cohomological dimension $\dim_G X$ is the smallest integer n such that $\check{H}^{n+1}(X, A; G) = 0$ for all closed subsets A of X .

Ever since the introduction of the cohomological dimension by P.S.Alexandroff [A], it was clear that there is a similarity between dimension and cohomological dimension. However, until recently, the dimension was investigated using mostly set-theoretic methods, and the cohomological dimension was investigated using mostly algebraic methods. Let us recall the basic results of the covering dimension theory [En]:

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Basic results of dimension theory of separable metric spaces.

Theorem 1.1 (Hurewicz-Wallman [En], p. 90). $\dim X \leq n$ iff $S^n \in AE(X)$, i.e. any map $f : A \rightarrow S^n$ extends over X if A closed in X .

Theorem 1.2 (Urysohn-Menger [En], p. 45). $\dim(A \cup B) \leq \dim A + \dim B + 1$.

Theorem 1.3 (Urysohn-Tumarkin-Hurewicz [En], p. 45). If $\dim X = m + n + 1$, then there are subsets A and B of X such that $X = A \cup B$ and $\dim A \leq m$, $\dim B \leq n$.

Theorem 1.4 (Tumarkin [En], p. 45). If $\dim X = n$, then there is a completely metrizable space X' containing X and with $\dim X' = n$.

Theorem 1.5 (Hurewicz [En], p. 61). If $\dim X = n$, then there is a compact metrizable space X' containing X and with $\dim X' = n$.

Theorem 1.6 (Menger [En], p. 46). $\dim(X \times Y) \leq \dim X + \dim Y$.

Theorem 1.7 (Nobeling [En], p. 120). There is a universal separable metrizable space of a given dimension.

Theorem 1.8 (Lefschetz [En], p. 126). There is a universal compact metrizable space of a given dimension.

Since $\dim_{\mathbf{Z}} X \leq n$ can be characterized as $K(\mathbf{Z}, n) \in AE(X)$, it was natural to seek analogs of the above statements for the cohomological dimension. So far, only the existence of universal compact spaces of a given integral cohomological dimension has eluded researchers in the field. Here are the analogs of 1.1-1.8 for the integral cohomological dimension:

Main results of integral dimension theory of separable metric spaces.

Theorem 1.9 (Cohen [Co]). $\dim_{\mathbf{Z}} X \leq n$ iff $K(\mathbf{Z}, n) \in AE(X)$, i.e., any map $f : A \rightarrow K(\mathbf{Z}, n)$ extends over X if A closed in X .

Theorem 1.10 (Rubin [Ru]). $\dim_{\mathbf{Z}}(A \cup B) \leq \dim_{\mathbf{Z}} A + \dim_{\mathbf{Z}} B + 1$.

Theorem 1.11 (Dranishnikov [Dra₁]). There is a compactum X such that $\dim_{\mathbf{Z}} X = 3$, and $X \neq A \cup B$ with $\dim_{\mathbf{Z}} A \leq 1$, $\dim_{\mathbf{Z}} B \leq 1$.

Theorem 1.12 (Rubin-Schapiro [R-S]). If $\dim_{\mathbf{Z}} X = n$, then there is a completely metrizable space X' containing X and with $\dim_{\mathbf{Z}} X' = n$.

Theorem 1.13 (Dydak-Walsh [D-W₁]). There is a separable metrizable space X such that $\dim_{\mathbf{Z}} X = 4$ and $\dim_{\mathbf{Z}} X' > 4$ for any compact space X' containing X .

Theorem 1.14 (Rubin-Schapiro [R-S]). $\dim_{\mathbf{Z}}(X \times Y) \leq \dim_{\mathbf{Z}} X + \dim_{\mathbf{Z}} Y$.

Theorem 1.15 (Dydak-Mogilski [D-M]). There is a universal separable metrizable space of a given integral cohomological dimension.

Question 1.16. Is there is a universal compact metrizable space of a given integral cohomological dimension ?

With the exception of 1.9, all of the above results were proved with difficulty exceeding similar results for the covering dimension. Especially interesting is the history of generalizing the Urysohn-Menger Theorem (see [Ru], [D-W₂], [D-R-S₁], [D-R], [Dy₂] and [Dy₃]).

In 1992, during a workshop on cohomological dimension theory in Knoxville, TN, A.Dranishnikov presented certain results (see [Dra₄]) of extraordinary dimension theory, which he called “Extension Theory”. Roughly speaking, one can create a new equivalence of compacta: $X \sim Y$ iff $K \in AE(X)$ is equivalent to $K \in AE(Y)$ for any CW complex K . Dually, one can introduce a new equivalence of CW complexes: $K \sim L$ iff $K \in AE(X)$ is equivalent to $L \in AE(X)$ for any compactum X . Dranishnikov’s idea was that Extension Theory ought to investigate these equivalences. Inspired by his talk, J.Dydak thought that perhaps Extension Theory could be a theory encompassing both the covering and the cohomological dimension theories. Since at that time J.Dydak was working on Kuzminov’s question “Does $\dim_G(A \cup B) \leq \dim_G A + \dim_G B + 1$ hold?”, it was natural to try to translate:

$$\dim(A \cup B) \leq \dim A + \dim B + 1$$

into the language of Extension Theory. A simple translation is:

$$\text{if } S^m \in AE(A) \text{ and } S^n \in AE(B), \text{ then } S^{m+n+1} \in AE(A \cup B).$$

What is the connection between S^m , S^n , and S^{m+n+1} ?

The answer is: S^{m+n+1} is the join of S^m and S^n .

Using this observation, J.Dydak [Dy₂] proposed the following (either as specific conjectures or problems which subsequently were converted into conjectures in [D-M]):

Candidates for basic results of Extension Theory of separable metric spaces.

Conjecture 1.17. *If $K \in AE(A)$ and $L \in AE(B)$ are CW complexes, then $K * L \in AE(A \cup B)$.*

Conjecture 1.18. *If $K * L \in AE(X)$, then there are subsets A, B of X such that $X = A \cup B$, $K \in AE(A)$, and $L \in AE(B)$.*

Conjecture 1.19. *If $K \in AE(X)$ is a countable CW complex, then there is a completion \bar{X} of X with $K \in AE(\bar{X})$.*

Conjecture 1.20. *Let K be a countable CW complex. Any X with $K \in AE(X)$ admits a compactification \bar{X} with $K \in AE(\bar{X})$ iff K is homotopy dominated by a finite polyhedron.*

Conjecture 1.21. *If K, L are CW complexes and $K \in AE(X)$, $L \in AE(Y)$, then $K \wedge L \in AE(X \times Y)$.*

Conjecture 1.22. *Suppose K is a countable CW complex. There is a universal separable space in the class $\{X \mid K \in AE(X)\}$.*

Conjecture 1.23. *Suppose K is a CW complex. The class $\{X \mid K \in AE(X)\}$ has a universal compact space iff K is homotopy dominated by a finite polyhedron.*

Several of the conjectures above were subsequently answered positively:

- a. Conjecture 1.17 by J.Dydak [Dy₂] for arbitrary metrizable spaces A, B and arbitrary CW complexes K, L .
- b. Conjecture 1.18 by A.Dranishnikov [Dra₂] for compact spaces X .
- c. Conjecture 1.19 by W.Olszewski [Ol₁].
- d. Conjecture 1.21 by A.Dranishnikov [Dra₅] in the special case $Y = [0, 1]$ and $L = S^1$.

- e. Conjecture 1.22 by W.Olszewski [Ol₂]. See also related work by M.Zarichnyi [Z].

Not only do the results of Extension Theory have simpler proofs than their predecessors in cohomological dimension theory; they give an explanation of certain phenomena in cohomological dimension theory and, sometimes, are more general than expected. Let us demonstrate this in the case of Dranishnikov's [Dra₂] result:

Theorem 1.24. *Suppose X is a compactum and K, L are countable CW complexes. If $K * L \in AE(X)$, then there are subsets A, B of X such that $X = A \cup B$, $K \in AE(A)$, and $L \in AE(B)$.*

He applied it successfully to the Mapping Intersection Problem in codimensions different from two. It turns out that Theorem 1.24 also implies Dranishnikov's Realization Theorem [Dra₁] which gives necessary and sufficient conditions for existence of compacta with prescribed values of cohomological dimension for each basic group. In the process of proving Theorem 1.24, Dranishnikov generalized a useful result from classical dimension theory:

Theorem 1.25 (Eilenberg-Borsuk [Ei], [Bor]). *If X is a separable metrizable space of dimension n , then for any map $f : A \rightarrow S^k$, A closed in X and $k < n$, there is an extension $f' : U \rightarrow K$ of f over an open set U such that $\dim(X - U) \leq n - k - 1$.*

Here is that generalization:

Theorem 1.26 (A.Dranishnikov [Dra₂]). *Let K, L be countable CW complexes. If X is a compactum and $K * L$ is an absolute extensor of X , then for any map $f : A \rightarrow K$, A closed in X , there is an extension $f' : U \rightarrow K$ of f over an open set U such that $L \in AE(X - U)$.*

The purpose of this paper is to extend results of the Extension Theory of compacta to separable metrizable spaces, to generalize some results of dimension theory, and to provide a partial answer to Conjecture 1.21. In particular, we will generalize the following well-known result:

Theorem 1.27 ([En], p. 41). *If X is a separable metrizable space of finite dimension $\dim X = n$, then for all $k < n$ there is a closed subset Y of X of dimension k .*

One may say that instead of studying extension types as in [Dra₇], we are trying to see if, given $K \in AE(X)$, $L \in AE(Y)$, and given a binary operation $R(\cdot, \cdot)$ on metrizable spaces (for example, $R(X, Y) = X \cup Y$ or $R(X, Y) = X \times Y$), one can match R with a binary operation S on CW complexes so that $S(K, L) \in AE(R(X, Y))$.

When translating results from Extension Theory to Cohomological Dimension Theory we find that the following theorem is of fundamental importance:

Theorem 1.28. *Suppose X is a metrizable space and K is a connected CW complex. Consider the following conditions:*

1. $K \in AE(X)$.
2. $SP^\infty(K) \in AE(X)$ ($SP^\infty(K)$ is the infinite symmetric product of K —see [D-T]).
3. $\dim_{H_m(K; \mathbf{Z})} X \leq m$ for all $m \geq 0$.
4. $\dim_{\pi_m(K; \mathbf{Z})} X \leq m$ for all $m \geq 0$.

Then, Condition 1) implies Condition 2). If K is simply connected, then Conditions 2-3-4 are equivalent. If X is of finite dimension and K is simply connected, then Conditions 1-2-3-4 are equivalent.

For X compact, Theorem 1.28 is due to A.Dranishnikov [Dra₃]. Subsequently, it has been generalized to metrizable spaces by J.Dydak [Dy₁], [Dy₂].

2. PRELIMINARY RESULTS

We need the following improvement of Tietze Extension Theorem:

Lemma 2.0. *Suppose $f : A \rightarrow [0, 1]$ is continuous and A is a closed subset of a metrizable space X . There is an extension $F : X \rightarrow [0, 1]$ of f such that $F^{-1}(0) = f^{-1}(0)$ and $F^{-1}(1) = f^{-1}(1)$.*

Proof. Choose a decreasing sequence of open neighborhoods U_n of $f^{-1}(0)$ in X whose intersection is $f^{-1}(0)$. Choose a decreasing sequence of open neighborhoods V_n of $f^{-1}(1)$ in X whose intersection is $f^{-1}(1)$. Given n , extend $f|_{A - (U_n \cup V_n)} : A - (U_n \cup V_n) \rightarrow (0, 1)$ to $g_n : X - (U_n \cup V_n) \rightarrow (0, 1)$. Extend $g_n \cup f$ to $f_n : X \rightarrow [0, 1]$. Now, define $F : X \rightarrow [0, 1]$ as $\sum_{n=1}^{\infty} f_n / 2^n$. \square

In most of the results of this paper, K, L are CW complexes. Occasionally, however, we will switch to simplicial complexes with the metric topology K_m . Obviously, K equipped with the CW topology contains more open sets, and therefore it is easier to construct maps into K_m rather than into K . On the other hand, given a map into a CW complex, one can take advantage of restrictions it imposes on the map (see Parts 6–7 of Lemma 2.1 below) and get more elegant proofs of some results. From the homotopy point of view there is no difference between K_m and K , and the purpose of the following lemma is to put together various results scattered throughout the literature; results which support this claim.

Lemma 2.1. *Suppose X is a metrizable space, and K is either a CW complex or a simplicial complex with the metric topology. Then:*

1. $K \in \text{ANE}(X)$, i.e., and if $f : A \rightarrow K$ is a map and A is a closed subset of X , then there are an open neighborhood U of A in X and a map $F : U \rightarrow K$ so that $F|_A = f$.
2. (**Homotopy Extension Theorem**) Any $f : A \times [0, 1] \cup X \times \{0\} \rightarrow K$, where A is a closed subset of X , extends over $X \times [0, 1]$.
3. If $f : A \rightarrow K$ is a map, $A \subset X$, and \mathcal{U} an open cover of a metric simplicial complex K , then there are an open neighborhood U of A in X , a map $F : U \rightarrow K$ and a homotopy $H : A \times [0, 1] \rightarrow K$ joining f and $F|_A$ such that each path $H(\{x\} \times I)$ of H is contained in an element of \mathcal{U} . Moreover, if $B \subset A$ is a closed subset of X , then one can construct H so that $H(\{x\} \times I)$ is a one-point set for all $x \in B$.
4. K is homotopy equivalent to a simplicial complex equipped with the metric topology.
5. If K is a simplicial complex, then the identity map $i : K \rightarrow K_m$ is a homotopy equivalence.
6. If K is a CW complex, then for any map $f : X \rightarrow K$ and for any $x \in X$ there is a neighborhood U of x such that $f(U)$ is contained in a finite subcomplex of K .

7. If K is a CW complex and if X is separable, then for any map $f : X \rightarrow K$, $f(X)$ is contained in a countable subcomplex of K .
8. If K is a metric simplicial complex, then $f : X \rightarrow K$ is continuous iff $\phi_v f$ is continuous for all vertices v of K ($\{\phi_v(x)\}$ are barycentric coordinates of $x = \sum_{v \in K^{(0)}} \phi_v(x) \cdot v$).

Proof. 1. If K is a CW complex, see [Ko]. If K is a metric simplicial complex, see [M-S], Theorem 11 on p.304.

2. See [M-S], Theorem 9 on p.41.

3. If K is a simplicial complex with the metric topology, one can use [M-S], Corollary 3 and Remark 3 on p.81.

4. See [M-S], Theorem 5 on p.317.

5. See [M-S], Theorem 10 on p.302.

6. Assume that $f(V)$ is not contained in a finite subcomplex of K for any neighborhood V of x in X . For each n choose, by induction, x_n so that $0 < \text{dist}(x, x_n) < 1/n$ and $f(x_n)$ does not belong to the smallest subcomplex of K containing $f(x)$ and all $f(x_k)$, $k < n$. The set $S = \{f(x_n)\}_{n>0}$ is a closed subset of K , which contradicts continuity of f .

7. Each $x \in X$ has a neighborhood V such that $f(V)$ is contained in a finite subcomplex of K . Now, X can be covered by a countable family of V_i such that each $f(V_i)$ is contained in a countable subcomplex of K .

8. See [M-S], Theorem 8 on p.301. \square

Definition 2.2. Given two CW complexes K and L , their join $K * L$ is formed as the adjoining space $K \oplus L \cup_{pr} K \times L \times [0, 1]$, where $pr : K \times L \times \{0, 1\} \rightarrow K \oplus L$ is defined by

$$pr(a, b, t) = \begin{cases} a & \text{if } t = 0, \\ b & \text{if } t = 1. \end{cases}$$

Notice that $K * L$ is homeomorphic to the union $M(\pi_1) \cup M(\pi_2)$ of mapping cylinders of projections $\pi_1 : K \times L \rightarrow K$ and $\pi_2 : K \times L \rightarrow L$, where we assume $M(\pi_1) \cap M(\pi_2) = K \times L$. Also, it is clear that this definition can be used to define the join of any two topological spaces.

Notice that there are canonical projections $\pi : K * L \rightarrow [0, 1]$, $\pi_K : K * L - L \rightarrow K$ and $\pi_L : K * L - K \rightarrow L$.

Definition 2.3. Given two simplicial complexes K and L , their simplicial join $K * L$ is formed by declaring $\langle s_0, \dots, s_k, t_0, \dots, t_l \rangle$ to be its simplex iff $\langle s_0, \dots, s_k \rangle$ is a simplex in K and $\langle t_0, \dots, t_l \rangle$ is a simplex in L . It carries a natural CW structure; its cells are of the form $s * \sigma$, where s is a simplex of K and σ is a simplex of L . Notice that $K * L$ with the CW topology is homeomorphic to the union $M(\pi_K) \cup M(\pi_L)$ of mapping cylinders of projections $\pi_K : K \times L \rightarrow K$ and $\pi_L : K \times L \rightarrow L$, where we assume $M(\pi_K) \cap M(\pi_L) = K \times L$. Thus, the simplicial join is related to the abstract join. As in the case of the abstract join, there are canonical projections $\pi : K * L \rightarrow [0, 1]$, $\pi_K : K * L - L \rightarrow K$ and $\pi_L : K * L - K \rightarrow L$. Namely, any point $x \in K * L$ can be expressed as $t \cdot u + (1 - t) \cdot v$, where $t \in [0, 1]$, $u \in K$, and $v \in L$. t is unique, and we put $\pi(x) = t$. u is unique if $x \notin L$, and we put $\pi_K(x) = u$. Similarly, v is unique if $x \notin K$, and we put $\pi_L(x) = v$.

The following lemma is crucial in all constructions involving maps to joins of metric simplicial complexes.

Lemma 2.4. *Suppose K, L are metric simplicial complexes and let $\pi : K * L \rightarrow [0, 1]$, $\pi_K : K * L - L \rightarrow K$, and $\pi_L : K * L - K \rightarrow L$ be canonical projections. If X is a metrizable space, then there is a one-to-one correspondence between maps $f : X \rightarrow K * L$ and 5-tuples $(U, V, g : U \rightarrow K, h : V \rightarrow L, \alpha : X \rightarrow [0, 1])$ satisfying the following conditions:*

1. $X = U \cup V$, U and V open in X ;
2. $\alpha^{-1}[0, 1) = U$ and $\alpha^{-1}(0, 1] = V$.

*Namely, given $f : X \rightarrow K * L$, define $U = f^{-1}(K * L - L)$, $V = f^{-1}(K * L - K)$, $g = \pi_K \circ f$, $h = \pi_L \circ f$ and $\alpha = \pi \circ f$. Conversely, given $(U, V, g : U \rightarrow K, h : V \rightarrow L, \alpha : X \rightarrow [0, 1])$, one defines f as follows:*

$$f(x) = \begin{cases} g(x) & \text{if } x \in U - V, \\ (1 - \alpha(x)) \cdot g(x) + \alpha(x) \cdot h(x) & \text{if } x \in U \cap V, \\ h(x) & \text{if } x \in V - U. \end{cases}$$

*The map f defined as above will be denoted by $g *_{\alpha} h$.*

Proof. The only thing we need to verify is that $f|_{U \cup V}$ is continuous. Notice that each point x of a simplicial complex M can be uniquely written as

$$x = \sum_{v \in M^{(0)}} \phi_v(x) \cdot v,$$

where $M^{(0)}$ is the set of vertices of M ($\{\phi_v(x)\}$ are called *barycentric coordinates* of x). To prove the continuity of f we need to show that $\phi_v f$ is continuous for all vertices v of $K * L$ (see [M-S], Theorem 8 on p.301, or Part 8 of Lemma 2.1). Without loss of generality we may assume that $v \in K^{(0)}$. Then,

$$\phi_v f(x) = (1 - \alpha(x)) \cdot \phi_v g(x) \text{ for all } x \in U$$

and

$$\phi_v f(x) = 0 \text{ for all } x \in U - V.$$

Clearly, $\phi_v f|_U$ is continuous. If $x_0 \in V - U$ and $x_n \rightarrow x_0$, $x_n \in U$, then $\alpha(x_n) \rightarrow 1$ and $0 \leq \phi_v g(x_n) \leq 1$. Consequently, $\phi_v f(x_n) \rightarrow 0 = \phi_v f(x_0)$. \square

Lemma 2.5. *Suppose K, L are metric simplicial complexes, and Y is a metrizable space such that $K * L \in AE(X)$ for a subset X of Y . Given closed subsets A of X and B of Y , and given maps $f : A \rightarrow K$, $g : B \rightarrow L$, there are a neighborhood U of $A \cup X$ in Y and a map $F : U \rightarrow K * L$ such that $F(A) \subset K * L - L$, $F(B) \subset K * L - K$, $F \circ \pi_K|_A = f$ and $F \circ \pi_L|_B \simeq g$, where $\pi_K : K * L - L \rightarrow K$ and $\pi_L : K * L - K \rightarrow L$ are projections.*

Proof. Extend f up to homotopy to $f' : V \rightarrow K$, and extend g up to homotopy to $g' : W \rightarrow L$ for some neighborhood V of A in Y and some neighborhood W of B in Y (see Part 3 of Lemma 2.1). Choose a map $\alpha : V \cup W \rightarrow [0, 1]$ such that $\alpha^{-1}(0) = V - W$ and $\alpha^{-1}(1) = W - V$. Now, use Lemma 2.4 and produce $G : V \cup W \rightarrow K * L$. Since $X \cap (A \cup B)$ is a closed subset of X , there is an extension $H : A \cup X \rightarrow K * L$ of $G|_{X \cap (A \cup B)}$. Let $\mathcal{U} = \{K * L - L, K * L - K\}$ be the open cover of $K * L$. Using Part 3 of Lemma 2.1, we can find an open neighborhood U of $A \cup X$ and an extension $F : U \rightarrow K * L$ of $F|_A$, such that $F|_{A \cup X}$ is homotopic to H in such a way that the paths of homotopy are contained in \mathcal{U} . It is clear that F is the required map. \square

The following lemma is useful when discussing generalizations of the Eilenberg-Borsuk Theorem.

Lemma 2.6. *Suppose K, L are metric simplicial complexes and X is a metrizable space. Let v be the vertex of the cone $\text{Cone}(K)$ of K . The following conditions are equivalent:*

1. *For any map $f : A \rightarrow K$, A closed in $Y \subset X$, there is an extension $f' : U \rightarrow K$ of f over an open set U of Y such that $L \in \text{AE}(Y - U)$.*
2. *For any map $f : A \rightarrow K$, A closed in $Y \subset X$, there is an extension $F : Y \rightarrow \text{Cone}(K)$ of $A \rightarrow K \hookrightarrow \text{Cone}(K)$ such that $L \in \text{AE}(F^{-1}(v))$.*
3. *For any map $f : A \rightarrow \text{Cone}(K)$, A closed in $Y \subset X$, there is an extension $F : Y \rightarrow \text{Cone}(K)$ of f such that $L \in \text{AE}(F^{-1}(v) - A)$.*

Proof. Let $\pi_K : \text{Cone}(K) - \{v\} \rightarrow K$ and $\pi : \text{Cone}(K) \rightarrow [0, 1]$ be projections.

1 \rightarrow 2. Notice that $\text{Cone}(K)$ is homeomorphic to the join $K * v$. Take an extension $f' : U \rightarrow K$ of f over an open set U such that $L \in \text{AE}(Y - U)$. Choose a map $\alpha : Y \rightarrow [0, 1]$ such that $\alpha^{-1}(0) = A$ and $Y - U = \alpha^{-1}(1)$. Put $V = \alpha^{-1}(0, 1]$, $g' : V \rightarrow \{v\}$, and use Lemma 2.4.

2 \rightarrow 3. Choose an extension $g : U \rightarrow K$ of $\pi_K \circ f : A - f^{-1}(v) \rightarrow K$ such that $L \in \text{AE}(Y - f^{-1}(v) - U)$ and U is open in $Y - f^{-1}(v)$. According to Lemma 2.0 there is an extension $H : Y \rightarrow [0, 1]$ of $h = \pi \circ f$ so that $H^{-1}(1) = Y - U$. Finally, use Lemma 2.4, or define $F : Y \rightarrow \text{Cone}(K)$ by

$$F(x) = \begin{cases} [g(x), H(x)] & \text{if } x \in U, \\ v & \text{if } x \in Y - U. \end{cases}$$

3 \rightarrow 1. Extend $f : A \rightarrow \text{Cone}(K)$ to $F : Y \rightarrow \text{Cone}(K)$ so that $L \in \text{AE}(F^{-1}(v) - A)$. Notice that $F^{-1}(v) - A = F^{-1}(v)$ and put $U = Y - F^{-1}(v)$, $f' = \pi_K \circ F : U \rightarrow K$. \square

Theorem 2.7. *Let K, L be CW complexes, and let X be a metrizable space. If, for any map $f : A \rightarrow K$, $A \subset X$, there is an extension $f' : U \rightarrow K$ of f over an open set U of $Y = X - (\text{cl}_X(A) - A)$ such that $L \in \text{AE}(Y - U)$, then $K * L$ is an absolute extensor of X .*

Proof. Since, by Part 2 of Lemma 2.1, being an absolute extensor is a property of the homotopy type, we may assume that both K and L are metric simplicial complexes. Suppose C is a closed subset of X , and $g : C \rightarrow K * L$ is a map. Let $A = C - g^{-1}(L)$, and let $f : A \rightarrow K$ be defined by $f(x) = \pi_K(g(x))$ for $x \in A$. Choose an extension $f' : U \rightarrow K$ of f over a neighborhood U of A in $Y = X - (\text{cl}_X(A) - A)$ so that $L \in \text{AE}(Y - U)$. Notice that $(Y - U) \cup g^{-1}(L)$ is closed in X , and the map $\pi_L \circ g : C - g^{-1}(K) \rightarrow L$ can be extended to $h : (Y - U) \cup C - g^{-1}(K) \rightarrow L$. Now, extend h to $h' : V \rightarrow L$ for some neighborhood V of $(Y - U) \cup C - g^{-1}(K)$ in $X - g^{-1}(K)$. Let $\pi : K * L \rightarrow [0, 1]$ be the projection. Extend the composition $\pi \circ g$ over $(X - U) \cup (X - V) \cup C$ by sending $X - U$ to 1, and by sending $X - V$ to 0. Extend the resulting map to $\alpha : X \rightarrow [0, 1]$ so that $\alpha^{-1}(0) = X - V$ and $\alpha^{-1}(1) = X - U$ (see Lemma 2.0). Now, use Lemma 2.4. \square

3. MAIN RESULTS

The following result is fundamental to section 3 of the paper. We arrived at it after analyzing Olszewski's paper [Ol₁], and we believe it offers a significant

simplification of his proof (see Theorem 3.6 in the present paper). To simplify its formulation we need the concept of a $\cup\cap$ -closed basis:

Definition 3.1. A family \mathcal{C} of closed subsets of the Hilbert cube Q is called a $\cup\cap$ -closed basis provided the family $\{\text{Int}_Q D \mid D \in \mathcal{C}\}$ is a basis of Q , \mathcal{C} is closed under finite unions and finite intersections, and $Q - \text{Int}_Q A \in \mathcal{C}$ if $A \in \mathcal{C}$.

Theorem 3.2. Let K be a CW complex and let \mathcal{C} be a $\cup\cap$ -closed basis of Q . Suppose \mathcal{F} is a non-empty family of triples (f, S, D) such that $S \subset Q$, $D \in \mathcal{C}$, and $f : S \cup D \rightarrow K$ is a map. Let $X = \bigcap_{(f, S, D) \in \mathcal{F}} S$. $K \in \text{AE}(X)$ if the following two conditions are satisfied:

1. $(f|(S' \cap S) \cup C, S' \cap S, C) \in \mathcal{F}$ if $(f, S, D) \in \mathcal{F}$, $(f', S', D') \in \mathcal{F}$, and $C \subset D' \cap D$, $C \in \mathcal{C}$.
2. Given $(f, S, D) \in \mathcal{F}$ and given an extension $g : (S \cap B) \cup D \cup C \rightarrow K$ of $f|(S \cap B) \cup D$, where $B, C \in \mathcal{C}$ with $B \cap C = \emptyset$, there is $(f', S', D \cup C) \in \mathcal{F}$ so that $S' \subset S$ and $g|(S' \cap B) \cup D \cup C \simeq f'|(S' \cap B) \cup D \cup C$.

Proof. Define a new family \mathcal{G} of triples (f, S, D) as follows: $(g, S, D) \in \mathcal{G}$ iff $g : S \cup D \rightarrow K$ is a map and there is $(f, S, D) \in \mathcal{F}$ with f homotopic to g . \mathcal{G} has stronger properties than \mathcal{F} :

- 1'. $(f|(S' \cap S) \cup C, S' \cap S, C) \in \mathcal{G}$ if $(f, S, D) \in \mathcal{G}$ and $(f', S', D') \in \mathcal{G}$, and $C \subset D' \cap D$, $C \in \mathcal{C}$.
- 2'. Given $(f, S, D) \in \mathcal{G}$ and given an extension $g : (S \cap B) \cup D \cup C \rightarrow K$ of $f|(S \cap B) \cup D$, where $B, C \in \mathcal{C}$ with $B \cap C = \emptyset$, there is $(f', S', D \cup C) \in \mathcal{G}$ so that $S' \subset S$ and $g|(S' \cap B) \cup D \cup C \simeq f'|(S' \cap B) \cup D \cup C$.

Property 1') is obvious. To prove Property 2'), notice that $(S \cap B) \cup D \cup C$ is closed in $S \cup D \cup C$ and $(S \cap B) \cup D$ is closed in $(S \cap B) \cup D \cup C$. Since $(f, S, D) \in \mathcal{G}$, there is $(h, S, D) \in \mathcal{F}$ with $f \simeq h$. Since $g|(S \cap B) \cup D \simeq h|(S \cap B) \cup D$, there is $g' : (S \cap B) \cup D \cup C \rightarrow K$ homotopic to g so that g' is an extension of $h|(S \cap B) \cup D$. Pick $(g'', S', D \cup C) \in \mathcal{F}$ so that $S' \subset S$ and $g''|(S' \cap B) \cup D \cup C \simeq g'|(S' \cap B) \cup D \cup C$. By the Homotopy Extension Theorem, there is $(f', S', D \cup C) \in \mathcal{G}$ with $f'|(S' \cap B) \cup D \cup C = g|(S' \cap B) \cup D \cup C$.

To prove 3.2 it suffices to show that any map $g : A \rightarrow K$ extends over X up to homotopy if A is a closed subset of X . Since any map $g : A \rightarrow K$ extends up to homotopy over a neighborhood of A in Q (see Part 3 of Lemma 2.1), we need to show that given a closed subset A of X , and given a map $g : V \rightarrow K$, V being a neighborhood of A in Q , the map $g|_A$ extends over X . We will construct by induction the following objects:

- i. a decreasing sequence $A_n \in \mathcal{C}$,
- ii. increasing sequences $B_n \in \mathcal{C}$ and $D_n \in \mathcal{C}$,
- iii. elements $(g_n, S_n, A_n \cap B_n) \in \mathcal{G}$,

such that the following conditions are satisfied:

1. $\bigcap_{n=1}^{\infty} A_n = \text{cl}_Q A$,
2. $V = \bigcup_{n=1}^{\infty} B_n$,
3. $S_{n+1} \subset S_n$,
4. $g_{n+1}|_{S_{n+1} - \text{Int}_Q A_n} = g_n|_{S_{n+1} - \text{Int}_Q A_n}$,
5. $g_n|_{A_n \cap B_n} = g|_{A_n \cap B_n}$.

Notice that for any closed subset B of Q , the family $\{D \in \mathcal{C} \mid B \subset \text{Int } D\}$ is a basis of neighborhoods of B . Therefore, there is a sequence $A_n \in \mathcal{C}$ with $A_{n+1} \subset \text{Int } A_n$ and $\bigcap_{n=1}^{\infty} A_n = \text{cl}_Q A$. Choose $B \in \mathcal{C}$, $B \subset V$ with $g|B \simeq \text{const}$. Choose a sequence $B_n \in \mathcal{C}$ with $B_{n+1} \supset \text{Int } B_n$, $B_1 = B$, and $V = \bigcup_{n=1}^{\infty} B_n$. Pick $(f_1, S_1, Q) \in \mathcal{G}$ with $f_1|B = g|B$. This can be done by observing that $\emptyset, Q \in \mathcal{C}$ so, given $(f, S, D) \in \mathcal{F}$, we have $(f|S, S, \emptyset) \in \mathcal{F}$ and any map $h : (S \cap \emptyset) \cup B \cup Q \rightarrow K$ extending $g|B$ is an extension of $g|(S \cap \emptyset)$, which implies the existence of f_1 . Given $(g_n, S_n, A_n \cap B_n) \in \mathcal{G}$ so that $g_n|A_n \cap B_n = g|A_n \cap B_n$, define

$$h : (S_n - \text{Int}_Q A_n) \cup A_n \cap B_n \cup A_{n+1} \cap B_{n+1} \rightarrow K$$

as

$$h|(S_n - \text{Int}_Q A_n) \cup A_n \cap B_n = g_n|(S_n - \text{Int}_Q A_n) \cup A_n \cap B_n$$

and

$$h|A_{n+1} \cap B_{n+1} = g|A_{n+1} \cap B_{n+1}.$$

Since $(S_n - \text{Int}_Q A_n) = S_n \cap (Q - \text{Int}_Q A_n)$ and $Q - \text{Int}_Q A_n \in \mathcal{C}$, there is $(g_{n+1}, S_{n+1}, A_n \cap B_n \cup A_{n+1} \cap B_{n+1}) \in \mathcal{G}$ so that $S_{n+1} \subset S_n$ and

$$\begin{aligned} g_{n+1}|(S_{n+1} - \text{Int}_Q A_n) \cup A_n \cap B_n \cup A_{n+1} \cap B_{n+1} \\ = h|(S_{n+1} - \text{Int}_Q A_n) \cup A_n \cap B_n \cup A_{n+1} \cap B_{n+1}. \end{aligned}$$

Replace g_{n+1} with $g_{n+1}|S_{n+1} \cup A_{n+1} \cap B_{n+1}$.

The direct limit g' of $g_n|X$ extends $g|A$. Notice that $g'|X - A$ is continuous, as for each $x \in X - A$ there is $n > 0$ with $x \in X - A_n \subset X - \text{Int}_Q A_n$ and $g'|X - \text{Int}_Q A_n = g_n|X - \text{Int}_Q A_n$. Also, g' is continuous at points of A , as for each $a \in A$ there is $n > 0$ such that $a \in \text{Int}_Q(A_n \cap B_n)$ and

$$g'|X \cap (\text{Int}_Q(A_n \cap B_n)) = g|X \cap (\text{Int}_Q(A_n \cap B_n)).$$

□

Corollary 3.3. *Let K be a CW complex, let \mathcal{C} be a $\cup\cap$ -closed basis of Q , and let X be a subset of Q . Then $K \in \text{AE}(X)$ iff any map $g : (X \cap B) \cup D \rightarrow K$, where $B, D \in \mathcal{C}$, extends over $X \cup D$.*

Proof. Suppose $K \in \text{AE}(X)$ and $g : (X \cap B) \cup D \rightarrow K$ is a map, where $B, D \in \mathcal{C}$. Notice that $(X \cap B) \cup D$ is closed in $X \cup D$. Extend g over a closed neighborhood N of $(X \cap B) \cup D$ in $X \cup D$. Let $G : N \rightarrow K$ be such an extension. Extend $G|X \cap N$ to $h : X \rightarrow K$ and define $g' : X \cup D \rightarrow K$ as $g'|X = h$ and $g'|(X \cap B) \cup D = g$.

Consider the family of all triples (f, X, D) such that $f : X \cup D \rightarrow K$ is a map. This family satisfies the conditions of Theorem 3.2, which implies $K \in \text{AE}(X)$. □

In order to apply Theorem 3.2 we need the following construction:

Lemma 3.4. *Suppose L is a countable subcomplex of a CW complex K , $a : T \rightarrow L$ is a map from a subset T of Q , $X \subset T$, and \mathcal{C} is a $\cup\cap$ -closed basis of Q . If $K \in \text{AE}(X)$, then there is a countable family $\text{Exp}(a, X, L, K)$ of triples (f', S', D) and a countable subcomplex $K(a)$ of K such that the following conditions are satisfied:*

- a. $(a, T, \emptyset) \in \text{Exp}(a, X, L, K)$.
- b. If $(f', S', D) \in \text{Exp}(a, X, L, K)$, then $f' : S' \cup D \rightarrow K(a)$ is a map, $X \subset S'$, $D \in \mathcal{C}$, $f'(D) \subset L$, and S' is open in Q if T is open in Q .

- c. $(f|(S' \cap S) \cup C, S' \cap S, C) \in \text{Exp}(a, X, L, K)$ if $(f, S, D) \in \text{Exp}(a, X, L, K)$ and $(f', S', D') \in \text{Exp}(a, X, L, K)$, and $C \subset D' \cap D$, $C \subset \mathcal{C}$,
d. Given $(f, S, D) \in \text{Exp}(a, X, L, K)$ and given an extension

$$g : ((S \cap B) \cup D \cup C, C) \rightarrow (K(a), L)$$

of $f|(S \cap B) \cup D$, where $B, C \in \mathcal{C}$ with $B \cap C = \emptyset$, there is $(f', S', D \cup C) \in \text{Exp}(a, X, L, K)$ so that $S' \subset S$ and

$$g|((S' \cap B) \cup D \cup C, C) \simeq f'|((S' \cap B) \cup D \cup C, C)$$

in $(K(a), L)$.

Proof. Suppose $B, C, D \in \mathcal{C}$ and $B \cap C = \emptyset$. Suppose $b : S \cup D \rightarrow K$ is a map so that $b(D \cap C) \subset L$ and $X \subset S$. There are countably many extensions $b_n : C \rightarrow L$ of $b|D \cap C$ so that any extension $h : C \rightarrow L$ of $b|D \cap C$ is homotopic rel. $D \cap C$ (in L) to one of b_n . Define $f_n : (S \cap B) \cup D \cup C \rightarrow K$ as the extension of $b|(S \cap B) \cup D$ satisfying $f_n|C = b_n$. Since $B \cap C = \emptyset$, f_n exists. By Lemma 2.5 (used with one term in the join being empty) there are an open neighborhood S_n of $X \cup (S \cap B) \cup D \cup C$ in $S \cup D \cup C$ and an extension $F_n : S_n \rightarrow K$ of f_n . Replace S_n with $S \cap S_n$. Let $\alpha(b)$ be the collection of all $(F_n|S_n \cup D \cup C, S_n, D \cup C)$ with all possible choices of D, C , and B . Clearly, $\alpha(b)$ is countable. Let K_b be a countable subcomplex of K containing L and containing $F_n(S_n \cup D \cup C)$ for all n and all possible choices of D, C (see 2.1.7).

Next, given a countable family F of triples (f, S, D) , $\beta(F)$ is defined as all triples $(f|(S \cap S') \cup C, S \cap S', C)$ so that $(f, S, D) \in F$ and $(f', S', D') \in F$, and $C \subset D' \cap D$, $C \in \mathcal{C}$.

Define families \mathcal{F}_n and subcomplexes K_n , $n \geq 1$, by induction as follows:

1. $\mathcal{F}_1 = \{(a, T, \emptyset)\}$, $K_1 = K_a$.
2. $\mathcal{F}_{n+1} = \alpha(\mathcal{F}_n) \cup \beta(\mathcal{F}_n)$, $K_{n+1} = \bigcup_{(f, S, D) \in \mathcal{F}_n} K_f$.

Notice that $\text{Exp}(a, X, L, K) = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ and $K(a) = \bigcup_{n=1}^{\infty} K_n$ satisfy the desired conditions. \square

The following theorem is useful when applying results of this paper to arbitrary, not necessarily countable, CW complexes.

Theorem 3.5. *Let K be a CW complex, and let X be a separable, metrizable space. If $K \in \text{AE}(X)$ and L is a countable subcomplex of K , then there is a countable subcomplex M of K which contains L and is an absolute extensor of X .*

Proof. Assume X is a subset of the Hilbert cube Q . Let \mathcal{C} be a countable family of closed subsets of the Hilbert cube Q which is a $\cup \cap$ -closed basis (see 3.1). Let $M_1 = L$ and let \mathcal{U}_1 consist of a single triple (c, Q, \emptyset) , where $c : Q \rightarrow M_1$ is a constant map. Suppose M_n is a countable subcomplex of K , and \mathcal{U}_n is a countable set of triples (f, U, D) , where U is an open set in Q containing X , $D \in \mathcal{C}$, and $f : U \cup D \rightarrow M_n$ is a map. We create (using Lemma 3.4) a new countable family \mathcal{U}' as the union of all $\text{Exp}(f, X, M_n, K)$, $(f, U, D) \in \mathcal{U}_n$. Now, as in 3.4, we find a countable subcomplex M_{n+1} of K containing M_n and containing $K(f)$ for any $(f, U, D) \in \mathcal{U}'$. Define \mathcal{U}_{n+1} by $\mathcal{U}_{n+1} = \mathcal{U}' \cup \mathcal{U}_n$. By Theorem 3.2 (more details below), the intersection $X' = \bigcap_{(f, U, D) \in \mathcal{U}} U$, where $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$, satisfies $M \in \text{AE}(X')$ if $M = \bigcup_{n=1}^{\infty} M_n$.

Therefore, $M \in AE(X)$. Indeed, suppose $(f, U, D) \in \mathcal{U}$, $f : U \cup D \rightarrow M_n$ for some n , and $g : (U \cap B) \cup D \cup C \rightarrow M$ is an extension of $f|(U \cap B) \cup D$, where $D, C \in \mathcal{C}$ and $B \cap C = \emptyset$. Then $g(C)$ is a compact subset of M ; therefore $g(C) \subset M_m$ for some $m > n$. Since \mathcal{U}_{m+1} contains a subset of the form $Exp(f, X, M_m, K)$, there is $(f', U', D \cup C) \in \mathcal{U}_{m+1}$ so that $f'|(S \cap B) \cup D \cup C$ is homotopic to g in M_{m+1} . \square

Theorem 3.6 (Completion Theorem of W.Olszewski [Ol₁]). *Let K be a countable CW complex. If X is a separable metrizable space, and K is an absolute extensor of X , then there is a completion X' of X such that $K \in AE(X')$.*

Proof. Assume X is a subset of the Hilbert cube Q . Let \mathcal{C} be a countable family of closed subsets of the Hilbert cube Q which is a $\cup\cap$ -closed basis (see 3.1). Let \mathcal{U}_1 consist of a single triple (c, Q, \emptyset) , where $c : Q \rightarrow K$ is a constant map. Suppose \mathcal{U}_n is a countable set of triples (f, U, D) , where U is an open set in Q containing X , $D \in \mathcal{C}$, and $f : U \cup D \rightarrow K$ is a map. We create (using Lemma 3.4) a new countable family \mathcal{U}_{n+1} as the union of all $Exp(f, X, K, K)$, $(f, U, D) \in \mathcal{U}_n$. By Theorem 3.2 the intersection $X' = \bigcap_{(f,U,D) \in \mathcal{U}} U$, where $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$, satisfies $K \in AE(X')$. \square

Theorem 3.7 (Generalized Eilenberg-Borsuk Theorem). *Let L be a countable CW complex. If X is a separable metrizable space and $K * L$ is an absolute extensor of X for some CW complex K , then for any map $r : A \rightarrow K$, A closed in X , there is an extension $r' : U \rightarrow K$ of r over an open set U such that $L \in AE(X - U)$.*

Proof. Let $\pi_K : K * L - L \rightarrow K$, $\pi_L : K * L - K \rightarrow L$, and $\pi : K * L \rightarrow [0, 1]$ be projections.

Since, by Part 2 of Lemma 2.1, being an absolute extensor is a property of the homotopy type, we may assume that both K and L are metric simplicial complexes. Let $\mathcal{C} = \{C_i\}_{i \geq 1}$ be a countable family of closed subsets of the Hilbert cube Q which is a $\cup\cap$ -closed basis. Suppose A is a closed subset of X , and $r : A \rightarrow K$ is a map. Extend r over a closed neighborhood N_1 of A in X and let $f_1 : N_1 \rightarrow K$ be an extension of r . Put $M_1 = X - \text{Int}_X N_1$, and let $g_1 : M_1 \rightarrow L$ be a constant map.

Suppose we have a finite sequence of maps $f_i : N_i \rightarrow K$, $1 \leq i \leq n$, such that the following conditions are satisfied if $M_i = X - \text{Int}_X N_i$:

1. N_{i+1} is a closed neighborhood of N_i in X .
2. $f_{i+1}|_{N_i} \simeq f_i$.
3. There is a map $g_i : M_i \cup D_i \rightarrow L$, $D_i \in \mathcal{C}$.
4. There are maps $g_{i,j,k,m,r} : (M_i \cap C_j) \cup C_k \cup C_m \rightarrow L$, $j, k, m \geq 1$, such that $C_j \cap C_m = \emptyset$, $C_k \subset D_i$, $g_{i,j,k,m,r}|_{(M_i \cap C_j) \cup C_k} \simeq g_i|(M_i \cap C_j) \cup C_k$, and for any extension $g : (M_i \cap C_j) \cup C_k \cup C_m \rightarrow L$ of $g_i|(M_i \cap C_j) \cup C_k$, there is (i, j, k, m, r) such that $g_{i,j,k,m,r} \simeq g$.

Consider the set of all 5-tuples (i, j, k, r, m) , $i \leq n$, so that there is no $p > k$ with $C_k \cup C_m \subset D_p$ and

$$g_p|(M_p \cap C_j) \cup C_k \cup C_m \simeq g_{i,j,k,m,r}|(M_p \cap C_j) \cup C_k \cup C_m.$$

Among all those 5-tuples choose one minimizing the sum $i + j + k + r + m$. Put $g_{i,j,k,m,r} = g$, $B = C_j$, $C = C_m$. Notice that $(M_n \cap B) \cup C_k \cup C$ is closed in $Y = X \cup C_k \cup C$. By Lemma 2.5 there is a map $h : Y \rightarrow K * L$ such that $\pi_K \circ h|_{N_n} \simeq f_n$ and $\pi_L \circ h|(M_n \cap B) \cup C_k \cup C = g|(M_n \cap B) \cup C_k \cup C$. Let N_{n+1} be a closed neighborhood of $N_n \cup (X \cap h^{-1}(K))$ in X , $N_{n+1} \subset X \cap h^{-1}(K * L - L)$, and

let $M_{n+1} = X - \text{Int}_X(N_{n+1})$. Put $D_{n+1} = C_k \cup C$, $g_{n+1} = \pi_L \circ h|_{M_{n+1} \cup C \cup C_k}$: $M_{n+1} \cup C \cup C_k \rightarrow L$ and $f_{n+1} = \pi_K \circ h|_{N_{n+1}}$.

Using the above construction we create a countable family of triples $\{(g_i, M_i, D_i)\}$ so that $\{(g_i|_{M_i \cup D_i \cap C_j}, M_i, D_i \cap C_j)\}$ satisfies the hypotheses of Theorem 3.2. Thus, $Y = \bigcap_{i=1}^{\infty} M_i$ satisfies $L \in AE(Y)$. Put $U = X - Y$ and notice that $U = \bigcup_{i=1}^{\infty} N_i$, which implies that f extends over U . \square

Theorem 3.8. *Let K, L be countable CW complexes. If X is a separable metrizable space, and $K * L$ is an absolute extensor of X , then there is a subset Y of X such that $K \in AE(Y)$ and $L \in AE(X - Y)$.*

Proof. Let \mathcal{C} be a countable family of closed subsets of the Hilbert cube Q which is a $\cup\cap$ -closed basis. Let \mathcal{U}_1 consist of a single triple (f, X, Q) , where f is a constant map from Q to K . Given a countable family of triples $\mathcal{U} = \{(f, U, D)\}$ such that U is open in X , $D \in \mathcal{C}$, $f : U \cup D \rightarrow K$ is a map, and $L \in AE(X - U)$, we create a new countable family $\mathcal{U}' = \{(f', U', D')\}$ of triples with the property that, given any $B, C, D \in \mathcal{C}$, $B \cap C = \emptyset$ and given any $g : (U \cap B) \cup D \cup C \rightarrow K$ such that $g|(U \cap B) \cup D = f|(U \cap B) \cup D$, then there is $(f', U', D \cup C) \in \mathcal{U}'$ so that $U' \subset U$, $f'|(U' \cap B) \cup D \simeq g|(U' \cap B) \cup D$, and $L \in AE(X - U')$. This is done as in Lemma 3.4 with the help of the following observation: if $g : (U \cap B) \cup D \cup C \rightarrow K$ is a map, then we extend it first to $G : N \rightarrow K$ for some closed neighborhood N of $(U \cap B) \cup D \cup C$ in $U \cup D \cup C$. Since $N \cap U$ is closed in U and $K * L \in AE(U)$, Theorem 3.7 guarantees the existence of an open neighborhood U' of $N \cap U$ in U such that $G|_{N \cap U}$ extends to $h : U' \rightarrow K$ and $L \in AE(U - U')$. The map $f' : U' \cup D \cup C \rightarrow K$, $f'|_{U'} = h$, $f'|(U' \cap B) \cup D \cup C = g$, can be used as in 3.4 to add a new element to \mathcal{U}' . Notice that $L \in AE(X - U')$. Indeed, $U - U'$ can be expressed as a countable union of closed sets in X . Since $X - U' = (X - U) \cup (U - U')$, we are done.

Define \mathcal{U}_n for $n > 1$ by $\mathcal{U}_n = \mathcal{U}'_{n-1}$. The intersection Y of the domains of all maps in $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ satisfies $K \in AE(Y)$ by Theorem 3.2. Notice that $L \in AE(X - Y)$. \square

The following result is related to Theorem 2.7:

Theorem 3.9. *Let K, L be countable CW complexes and let X be a separable metrizable space. If for any map $f : A \rightarrow K$, A closed in an open subset U of X , there is an extension $f' : U' \rightarrow K$ so that $L \in AE(U - U')$, then $K * L$ is an absolute extensor of X .*

Proof. As in the proof of 3.8, we can find $Y \subset X$ so that $K \in AE(Y)$ and $L \in AE(X - Y)$. By Theorem 1.2 of [Dy₂] (see Conjecture 1.17), $K * L \in AE(X)$. \square

Theorem 3.10. *Let X be a separable metrizable space, and let K_i , $i \leq n$, be countable CW complexes. If, for all $i \leq n$, $\Sigma^2 K_i \in AE(X)$, $\Sigma K_i \notin AE(X)$, then there are a closed subset Y of X such that, for all $i \leq n$, $\Sigma K_i \in AE(Y)$, $K_i \notin AE(Y)$.*

Proof. Fix $j \leq n$. Since $S^0 * K_j = \Sigma K_j \notin AE(X)$, there is a closed subset A of U , U open in X , and a map $f : A \rightarrow S^0$ such that, for any extension $F : U' \rightarrow S^0$ of f over an open neighborhood U' of A in U , $K_j \notin AE(U - U')$ (see Theorem 3.9).

On the other hand, $S^0 * (\bigvee_{i=1}^n \Sigma K_i)$ is homotopy equivalent to $\bigvee_{i=1}^n \Sigma^2 K_i \in AE(U)$,

which implies, by Theorem 3.7, that there is an extension $F : U' \rightarrow S^0$ of f over an open neighborhood U' of A in U such that $\bigvee_{i=1}^n \Sigma K_i \in AE(U - U')$. Express $U - U'$ as the union of countably many closed sets $Y_{U,i}$ in X . There is k such that $K_j \notin AE(Y_{U,k})$. Put $Y_j = Y_{U,i}$. Finally, put $Y = \bigcup_{i=1}^n Y_i$. \square

Theorem 3.11. *Let X be a separable metrizable space, and let $K * L$ be an absolute extensor of X for some metric simplicial complexes K and L . If K and L are countable, then for any map $f : A \rightarrow K * L$, A closed in X , there is an extension $g : X \rightarrow K * L$ of f such that $K \in AE(g^{-1}(K) - A)$ and $L \in AE(g^{-1}(L) - A)$.*

Proof. Let $\pi : K * L \rightarrow K * \{v\}$ be the natural projection. Extend $\pi \circ f : A \rightarrow K * \{v\}$ to $F : X \rightarrow K * \{v\}$ so that $L \in AE(F^{-1}(v) - A)$ (see Lemma 2.5). Extend $\pi_L \circ f : f^{-1}(L) \rightarrow L$ to $G : F^{-1}(v) \rightarrow L$. Now, extend $G \cup f|(A - f^{-1}(K))$ to $H : U \rightarrow L$ for some open neighborhood U of $F^{-1}(v) \cup (A - f^{-1}(K))$ in $X - f^{-1}(K)$.

Let $pr : K * L \rightarrow \{v\} * L$ be the natural projection. Extend $pr \circ f : A \rightarrow \{v\} * L$ to $J : X \rightarrow \{v\} * L$ so that $K \in AE(J^{-1}(v) - A)$ (see Lemma 2.5). Extend $pr_K \circ f : f^{-1}(K) \rightarrow K$ to $P : J^{-1}(v) \rightarrow K$. Now, extend $P \cup f|(A - f^{-1}(L))$ to $R : V \rightarrow K$ for some open neighborhood V of $J^{-1}(v) \cup (A - f^{-1}(L))$ in $X - f^{-1}(L)$.

Let $\pi : K * L \rightarrow [0, 1]$ be the projection. Extend the composition $\pi \circ f$ over $(X - U) \cup (X - V) \cup A$ by sending $X - U$ to 0, and by sending $X - V$ to 1. Extend the resulting map to $\alpha : X \rightarrow [0, 1]$ so that $\alpha^{-1}(1) = X - V$ and $\alpha^{-1}(0) = X - U$ (see Lemma 2.0). Now, use Lemma 2.4. \square

Corollary 3.12. *Let X be a separable metrizable space, and let K be a countable metric simplicial complex. For any map $f : A \rightarrow Cone(K)$, A closed in X , there is an extension $g : X \rightarrow Cone(K)$ of f such that $K \in AE(g^{-1}(K) - A)$.*

4. APPLICATIONS TO COHOMOLOGICAL DIMENSION THEORY

Theorem 4.1. *Suppose G_1, \dots, G_n are countable, non-trivial, abelian groups, and $k > 0$. For any separable metrizable space X of finite dimension $\dim X > 0$, there is a closed subset Y of X with*

$$\dim_{G_i} Y = \max(\dim_{G_i} X - k, 1)$$

for $i = 1, \dots, n$.

Proof. It suffices to consider the case $k = 1$. Choose a closed subset A of X of dimension 1 (apply Theorem 3.10 or use Theorem 1.27). Let $k_i = \dim_{G_i} X$. Create a new list H_1, \dots, H_m consisting of groups G_i so that $k_i > 2$. Let $M_i(q)$ be the Moore space $M(H_i, q)$. For $i \leq m$, let $K_i = M(H_i, k_i - 2)$. By Theorem 3.10, there is a closed subset B of X such that $K_i \notin AE(B)$, but $\Sigma(K_i) \in AE(B)$ for all $i \leq m$. By Theorem 1.28, $\dim_{H_i} B = k_i - 1$ for all $i \leq m$. Put $Y = A \cup B$. \square

Corollary 4.2. *Suppose $G_i, i \geq 1$, are countable, non-trivial, abelian groups, and $k > 0$. For any separable metrizable space X of finite dimension $\dim X > 0$ there is a σ -closed subset Y of X such that*

$$\dim_{G_i} Y = \max(\dim_{G_i} X - k, 1)$$

for $i \geq 1$.

Proof. Let $F_i = \bigoplus_{j=i}^{\infty} G_j$. By Theorem 4.1 there is a closed subset Y_n of X such that $\dim_G Y_n = \max(\dim_G X - k, 1)$ for all $G \in \{G_1, \dots, G_n, F_n\}$. Let Y be the union of all Y_n , $n \geq 1$. \square

Theorem 4.3. *Suppose G_1, \dots, G_n are non-trivial, abelian groups and $k > 0$. For any σ -compact metrizable space X of finite dimension $\dim X > 0$, there is a compact subset Y of X with*

$$\dim_{G_i} Y = \max(\dim_{G_i} X - k, 1)$$

for $i = 1, \dots, n$.

Proof. By the Bockstein Theorem [Ku], for each abelian group G there is a countable abelian group H such that $\dim_G X = \dim_H X$ for all compacta X . Now, use Theorem 4.1. \square

Corollary 4.4. *For any σ -compact metrizable space X of finite dimension $\dim X > 0$, and for any $k > 0$, there is a σ -compact subset Y of X with*

$$\dim_G Y = \max(\dim_G X - k, 1)$$

for any abelian group G .

Proof. It suffices to consider compact X only. By the Bockstein Theorem [Ku] there is a sequence $\mathcal{BG} = \{B_1, \dots, B_n, \dots\}$ of countable abelian groups such that for each abelian group G there is a subset $\sigma(G)$ of \mathcal{BG} with the property that $\dim_G X = \sup\{\dim_H X \mid H \in \sigma(G)\}$ for all compacta X . Now, use Corollary 4.2. \square

Theorem 4.5. *Suppose G is a countable ring with unity, k is an integer, and X is a separable metrizable space of finite dimension. Then, $\dim_G X \leq n$, where $n \geq 2$, if and only if any map $f : A \rightarrow K(G, k)$, A closed in an open subset U of X , extends over a neighborhood U' of A in U with*

$$\dim_G(U - U') \leq n - k - 1.$$

Proof. Suppose any map $f : A \rightarrow K(G, k)$, A closed in an open subset U of X , extends over a neighborhood U' of A in U with $\dim_G(U - U') \leq n - k - 1$. By Theorem 3.9, $K(G, k) * K(G, n - k - 1) \in AE(X)$, and by Theorem 3.8 there is a subset Y of X with $\dim_G Y \leq k$ and $\dim_G(X - Y) \leq n - k - 1$, which implies $\dim_G X \leq n$ in view of Theorem 1.4 of [Dy2].

Suppose $\dim_G X \leq n$, and $f : A \rightarrow K(G, k)$, A closed in an open subset U of X . Notice that the homotopy groups of $K(G, k) * K(G, n - k - 1)$ are trivial in the range from 0 to $n - 1$, and are G -modules starting from n . Therefore, $K(G, k) * K(G, n - k - 1) \in AE(X)$ (see [Dy1], Theorems F-G) and, by Theorem 3.7, f extends over a neighborhood U' of A in U with $\dim_G(U - U') \leq n - k - 1$. \square

Theorem 4.6. *Suppose G is an abelian group, and X is a separable metrizable space with $\dim_G X < \infty$. Then, for any countable subgroup G' of G , there is a countable subgroup H of G containing G' such that $\dim_G X = \dim_H X$.*

Proof. Let $\dim_G X = n$. Arrange all countable subgroups of G in a direct system $\{G_s \mid h_{s,s'}, S\}$ so that $s < s'$ means that $G_s \subset G_{s'}$, and $h_{s,s'} : G_s \rightarrow G_{s'}$ is the inclusion. Let N be the simplicial complex whose set of vertices is S and $\langle s_0, \dots, s_m \rangle$ is an m -simplex of N iff $s_0 > \dots > s_m$. As in Lemma 2.6 of [Dy1],

N is contractible and there are a CW complex L of the type $K(G, n)$ and a map $\pi : L \rightarrow N$ such that for each simplex Δ of N , $\pi^{-1}(\Delta)$ is a countable subcomplex of L which is of type $K(G_s, n)$, where s is the minimal vertex of Δ . Moreover, the inclusion $\pi^{-1}(\Delta') \rightarrow \pi^{-1}(\Delta)$ induces $h_{s, s'} : G_s \rightarrow G_{s'}$ on the n -th homotopy groups. Given a countable subcomplex L' of L , one can find a vertex v of N and a countable subcomplex M of N so that $w > v$ for all vertices w of M , and $\pi(L') \subset M$. Now, $M * v$ is a subcomplex of N , $\pi^{-1}(M * v) \supset L'$, and $\pi^{-1}(M * v)$ is of type $K(G_v, n)$. Choose $v(1)$ so that $G_{v(1)} = G'$, and let $N_1 = \{v(1)\}$. Using induction, we can construct an increasing sequence N_i of countable subcomplexes of N , and an increasing sequence L_i of countable subcomplexes of L such that the following conditions are satisfied:

1. $\pi^{-1}(N_i) \subset L_i$.
2. $L_i \in AE(X)$.
3. $\pi(L_i) \subset N_{i+1}$.
4. $N_i * v_i \subset N_{i+1}$ for some vertex v_i of N_{i+1} .

Let $M = \bigcup_{i=1}^{\infty} N_i$. Notice that $\pi^{-1}(M) = \bigcup_{i=1}^{\infty} L_i \in AE(X)$ is of type $K(H, n)$ for some countable subgroup H of G which contains G' . \square

In the following two results $H_k(X; G)$ stands for the k -th reduced Steenrod homology group of X if X is compact, and stands for the k -th reduced Steenrod-Sitnikov homology group of X if X is not compact. (See [Mas] for the properties of Steenrod homology groups.)

The purpose of Theorem 4.7 is to expose difficulties one has to face when generalizing results of the cohomological dimension theory of compacta to separable, metrizable spaces. It was shown by Dranishnikov, Repovš, and Ščepin [D-R-S₂] that the Bockstein Theorem does not hold for separable, metrizable spaces. Our result is even stronger.

Theorem 4.7. *Suppose G is a countable abelian group, and A_p is the ring of p -adic integers for some prime number p . There is a separable space W of dimension 2 such that*

$$\dim_G W \neq \dim_{A_p} W.$$

Proof. If G is a torsion group or $G/\text{Tor } G$ is divisible by p , then there is a compactum W of dimension 2 such that $\dim_G W = 1$ and $\dim_{A_p} W = 2$ (see [Dra₁]). Assume $G/\text{Tor } G$ is not divisible by p . Express $(G/\text{Tor } G) \otimes \mathbf{Z}_{(p)}$ as $H \oplus D$, where D is a maximal divisible subgroup (a direct sum of copies of rationals) of $(G/\text{Tor } G) \otimes \mathbf{Z}_{(p)}$. Notice that $H \neq 0$, and $\bigcap_{n=1}^{\infty} p^n H = \{0\}$. By Lemma 4.8 there is a two-point subset D of the p -adic solenoid C such that $H_1(C/D; H) = 0$. Let X be a (C, D) -continuum of dimension 1, i.e. any map $f : A \rightarrow D$, A closed in X , extends to $f' : X \rightarrow C$ (see [D-R-S₂]). As in [D-R-S₂] (Propositions 1.3-1.5) one gets that, for every pair of closed subsets $B \subset A$ of X , the group $H_1(A/B; H)$ is trivial.

Let T be the subset of R^3 obtained by connecting every pair a, b of different points in Q^3 (Q being the rationals) by a copy X_{ab} of X so that the diameter of X_{ab} is $|a - b|$. We request that $X_{ab} \cap X_{cd} = \{a, b\} \cap \{c, d\}$ if $\{a, b\} \neq \{c, d\}$. Notice that $H_0(X; A_p) = \text{Ext}(H^1(X; \mathbf{Z}), A_p) = 0$. Therefore, by Proposition 2.6 of [D-R-S₂], $H_0(U \cap T; A_p) = 0$ for every open ball U of R^3 , and, by the Sitnikov

Duality, $H^2(U \cap (R^3 - T)/(\partial U \cap (R^3 - T)); A_p) = 0$. Thus, by Lemma 3.5 of [D-R-S₂], $\dim_{A_p} W = 1$. To show that $\dim_H W = 2$ it suffices to prove $H_0(T; H) \neq 0$. Suppose $H_0(T; H) = 0$, and choose $D = \{a, b\}$, $a \neq b \in Q^3$. There is a compact subset Y of T such that $H_0(D; H) \rightarrow H_0(Y; H)$ is not a monomorphism. There is an open subset V of $Y - D$ such that $V \subset X_{cd}$ for some $c, d \in Q^3$ (use the Baire Category Theorem). Since $H_1(Y/(Y - V); H) = 0$, it follows that $H_0(Y - V; H) \rightarrow H_0(Y; H)$ is a monomorphism, and $H_0(D; H) \rightarrow H_0(Y - V; H)$ is not a monomorphism. As in [D-R-S₂] (Lemma 3.3), one proves that there must be a minimal Y so that $H_0(D; H) \rightarrow H_0(Y; H)$ is not a monomorphism, and that Y must be D , a contradiction. By Corollary 2.8 of [Dy₁], and Theorem B of [Dy₁], $2 = \dim W \geq \dim_G W \geq \dim_H W = 2$, which completes the proof. \square

Lemma 4.8. *Suppose G is a countable, torsion-free abelian group such that*

$$\bigcap_{n=1}^{\infty} p^n G = \{0\}$$

for some prime number p . If Σ_p is the p -adic solenoid, then there is a two-point subset D of Σ_p such that $H_1(\Sigma_p/D; G) = 0$.

Proof. This lemma is a generalization of Proposition 1.1 of [D-R-S₂], which deals with the case $G = \mathbf{Z}$, and characterizes two-point subsets D of Σ_p with $H_1(\Sigma_p/D; \mathbf{Z}) = 0$ as those homotopic to a pair a, b with $b - a \in A_p - \mathbf{Z}_{(p)}$. Our plan is to use ideas of that proposition with the distinction that we have no characterization of two-point subsets D of Σ_p with $H_1(\Sigma_p/D; G) = 0$.

We will find $D = \{0, c\} \subset A_p$, where $c = n_0 + n_1p + \dots + n_kp^k + \dots$ and $0 \leq n_i < p$ for all i (see [Kob]). Notice that $H_1(\Sigma_p/D; G) = H_1(\Sigma_p \cup I; G) = \check{H}_1(\Sigma_p \cup I; G)$, where I is an interval joining 0 and c . $\Sigma_p \cup I$ is the inverse limit of $S^1 \cup I \leftarrow S^1 \cup I \leftarrow \dots$ with each bonding map being a homeomorphism on I and sending z to z^p on S^1 . As in [D-R-S₂] (where $G = \mathbf{Z}$), $\check{H}_1(\Sigma_p \cup I; G)$ is the inverse limit of $G \oplus G \leftarrow G \oplus G \leftarrow \dots$ and the i -th homomorphism $\phi_i^{i+1} : G \oplus G \rightarrow G \oplus G$ is given by $\phi_i^{i+1}(g, h) = (pg + n_ih, h)$. Hence, $\phi_0^{k+1}(g, h) = (p^{k+1}g + c_kh, h)$, where $c_k = n_0 + n_1p + \dots + n_kp^k$. Thus, $\text{Im } \phi_0^{k+1}$ can be characterized as pairs (u, v) with $p^{k+1}|u - c_kv$, and $\text{Im } \phi_0^\infty = \text{Im}(\check{H}_1(\Sigma_p \cup I; G) \rightarrow G \oplus G)$ consists of pairs (u, v) with $p^{k+1}|u - c_kv$ for all k .

Let us fix a pair $(u, v) \in G \oplus G - (0, 0)$, and let us show that there is at most one $c = n_0 + n_1p + \dots + n_kp^k + \dots \in A_p$ with $p^{k+1}|u - c_kv$ for all k . Indeed, suppose $d = m_0 + m_1p + \dots + m_kp^k + \dots \in A_p$ and $p^k|u - d_kv$ for all k . Choose the smallest i with $n_i \neq m_i$, and notice that $p^{j+1}|(u - c_jv) - (u - d_jv) = (d_j - c_j)v$ for all $j \geq i$. Since $d_j - c_j$ is not divisible by p^{i+1} , it follows that $p^{j+1-i}|v$. Thus, $v = 0$, and, since $p^{k+1}|u - c_kv$ for all k , it follows that $u = 0$. Since G is countable, there is a countable subset $S_0 \subset \Sigma_p$ such that if $c \in \Sigma_p - S_0$ then

$$\text{Im } \phi_0^\infty = \text{Im}(\check{H}_1(\Sigma_p \cup I; G) \rightarrow G \oplus G)$$

is trivial. Similarly, there is a countable subset $S_k \subset \Sigma_p$ such that if $c \in \Sigma_p - S_k$ then

$$\text{Im } \phi_k^\infty = \text{Im}(\check{H}_1(\Sigma_p \cup I; G) \rightarrow G \oplus G)$$

is trivial. Finally, choose $c \in A_p - \bigcup_{i=0}^\infty S_i$. \square

5. COHOMOLOGICAL DIMENSION OF CARTESIAN PRODUCTS

In 1991, E.Ščepin (private communication) speculated on the connection between $\dim(A*B)$ and $\dim(A \cup B)$. The idea was that there ought to be a similarity between the two expressions. In particular, he conjectured that

$$1 + \dim(A \times B) = \dim(A * B) \geq \dim(A \cup B).$$

This conjecture was partially confirmed by A.Dranishnikov [Dra₇]. Namely, he proved that if $X = A \cup B$ is a compactum, then $1 + \dim(A \times B) \geq \dim(A \cup B)$ if $\dim(X \times X) = 2 \dim X$, and $2 + \dim(A \times B) \geq \dim(A \cup B)$ if $\dim(X \times X) \neq 2 \dim X$. In 1992, J.Dydak [Dy₂] proved the following result, which was conceived as a Urysohn-Menger Theorem in cohomological dimension theory:

Theorem 5.1. *Suppose A and B are subsets of a metrizable space, and G and H are abelian groups. Then*

$$\dim_{G \otimes H}(A \cup B) \leq \dim_G A + \dim_H B + 1$$

and

$$\dim_{G*H}(A \cup B) \leq \dim_G A + \dim_H B + 2.$$

Theorem 5.1 was deduced from the analog of the Urysohn-Menger Theorem in extension theory: $K \in AE(A)$, $L \in AE(B)$ implies $K * L \in AE(A \cup B)$. In this paper, we reinterpret Ščepin's idea to mean that if there is an estimate of $\dim_G(A \cup B)$ using $\dim_G A$ and $\dim_G B$, then there ought to be a similar estimate of $\dim_G(A * B) = 1 + \dim_G(A \times B)$. Here is an analog of the Menger Theorem in cohomological dimension theory which we obtained using this reasoning:

Theorem 5.2. *Suppose X is a metrizable space, Y is a σ -compact metrizable space, and G, H are abelian groups. Then*

$$\dim_{G \otimes H}(X \times Y) \leq \dim_G X + \dim_H Y$$

and

$$\dim_{G*H}(X \times Y) \leq \dim_G X + \dim_H Y + 1.$$

Proof. Suppose $G, H \neq 0$ are abelian groups and $\dim_G X = m$, $\dim_H Y = n$.

We may assume Y is compact. First, let us prove that

$$\dim_{G*H}(X \times Y) \leq m + n + 1.$$

Since $G * H = (\text{Tor } G) * (\text{Tor } H)$ and $\dim_F Z \geq \dim_{\text{Tor } F} Z$ for any metrizable space Z (see [Dy₁], Theorem B), we may assume $G = \text{Tor } G$, $H = \text{Tor } H$. Since each torsion group is the direct limit of its \mathbf{p} -groups, and the torsion product commutes with the direct product, we may assume that G is a \mathbf{p} -group and H is a \mathbf{q} -group for some primes \mathbf{p}, \mathbf{q} . If $\mathbf{p} \neq \mathbf{q}$, then $G * H = 0$, and we are done. So, assume $\mathbf{p} = \mathbf{q}$. If $G * H$ is not divisible by \mathbf{p} , then either G or H is not divisible by \mathbf{p} . In such a case $\dim_{G*H}(X \times Y) = \dim_{\mathbf{Z}/\mathbf{p}}(X \times Y) \leq \dim_{\mathbf{Z}/\mathbf{p}} X + \dim_{\mathbf{Z}/\mathbf{p}} Y$ (see Theorem 22 of [Ku]), and, since $\dim_{\mathbf{Z}/\mathbf{p}} T \leq \dim_F T + 1$ for any metrizable space T and any \mathbf{p} -group F (see Section 4 of [Ku] or Theorem B of [Dy₁]), we are done. If $G * H$ is divisible by \mathbf{p} , then $\dim_{G*H}(X \times Y) = \dim_{\mathbf{Z}/\mathbf{p}^\infty}(X \times Y) \leq \dim_{\mathbf{Z}/\mathbf{p}^\infty} X + \dim_{\mathbf{Z}/\mathbf{p}^\infty} Y + 1$ (see Theorem 22 of [Ku]), and, since $\dim_{\mathbf{Z}/\mathbf{p}^\infty} T \leq \dim_F T$ for any metrizable space T and any \mathbf{p} -group F (see Section 4 of [Ku] or Theorem B of [Dy₁]), we are done.

Now, let us prove that $\dim_{G \otimes H}(X \times Y) \leq m + n$. As in Section 5 of [Ku], it suffices to show that $\dim_{H^k(Y/T; G \otimes H)} X \leq m + n - k$ for all closed subsets T of Y

and all k such that $H^k(Y/T; G \otimes H) \neq 0$. Since $\dim_{G \otimes H} Y \leq n$ (see Corollary 3.1 of [Dy₁]), $H^k(Y/T; G \otimes H) \neq 0$ implies $k \leq n$. By exactness of

$$0 \rightarrow H^k(Y/T; \mathbf{Z}) \otimes G \otimes H \rightarrow H^k(Y/T; G \otimes H) \rightarrow H^{k+1}(Y/T; \mathbf{Z}) * (G \otimes H) \rightarrow 0,$$

and in view of

$$\dim_{H^k(Y/T; \mathbf{Z}) \otimes G \otimes H} X \leq m \leq m + n - k$$

if $k \leq n$ (see Corollary 3.1 of [Dy₁]), it suffices to show that

$$\dim_{H^{k+1}(Y/T; \mathbf{Z}) * (G \otimes H)} X \leq m + n - k$$

if $k \leq n$. Let $F_{\mathbf{p}}$ be the \mathbf{p} -torsion of $F = H^{k+1}(Y/T; \mathbf{Z}) * (G \otimes H)$ for some prime \mathbf{p} . If G is not divisible by \mathbf{p} , then either its \mathbf{p} -torsion is not divisible by \mathbf{p} or $G/\text{Tor } G$ is not divisible by \mathbf{p} . In either case $m + n - k \geq m = \dim_G X \geq \dim_{\mathbf{Z}/\mathbf{p}} X \geq \dim_{F_{\mathbf{p}}} X$. Thus, it remains to consider the case $G = \mathbf{p} \cdot G$.

Case 1: $F_{\mathbf{p}}$ is not divisible by \mathbf{p} . Since G is divisible by \mathbf{p} , H cannot be divisible by \mathbf{p} as the \mathbf{p} -torsion of $G \otimes H$ would be trivial. Thus, the \mathbf{p} -torsion of $H^{k+1}(Y/T; \mathbf{Z})$ is not divisible by \mathbf{p} . Hence $H^{k+1}(Y/T; \mathbf{Z}) \otimes H \neq 0$, which implies $H^{k+1}(Y/T; H) \neq 0$. The consequence is that $k < \dim_H Y = n$. If $\mathbf{p}\text{-Tor } G = 0$, then $\mathbf{p}\text{-Tor}(G \otimes H) = 0$, and $F_{\mathbf{p}} = 0$. Thus, $\mathbf{p}\text{-Tor } G \neq 0$ and $m + n - k \geq m + 1 = \dim_G X + 1 \geq \dim_{\mathbf{Z}/\mathbf{p}^\infty} X + 1 \geq \dim_{F_{\mathbf{p}}} X$.

Case 2: $F_{\mathbf{p}} \neq 0$ is divisible by \mathbf{p} . In this case both groups $H^{k+1}(Y/T; \mathbf{Z})$ and $G \otimes H$ have non-trivial \mathbf{p} -torsions which are divisible by \mathbf{p} . Since $G = \mathbf{p} \cdot G$, its \mathbf{p} -torsion must be non-trivial and $m + n - k \geq m = \dim_G X \geq \dim_{\mathbf{Z}/\mathbf{p}^\infty} X = \dim_{F_{\mathbf{p}}} X$. \square

Theorem 5.3. *Suppose X is a metrizable space, and Y is a σ -compact metrizable space. Then*

$$\dim_{\mathbf{R}}(X \times Y) \leq \dim_{\mathbf{R}} X + \dim_{\mathbf{R}} Y$$

for any ring \mathbf{R} with unity, and

$$\dim_G(X \times Y) \leq \dim_G X + \dim_G Y + 1$$

for any abelian group G .

Proof. If \mathbf{R} is a ring with unity 1, then \mathbf{R} is a retract of $\mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R}$. Indeed, the homomorphism $m : \mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R} \rightarrow \mathbf{R}$ induced by the multiplication $\mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ is a left inverse of $id \otimes 1 : \mathbf{R} \rightarrow \mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R}$. Thus,

$$\dim_{\mathbf{R}}(X \times Y) \leq \dim_{\mathbf{R}} X + \dim_{\mathbf{R}} Y.$$

Let G be an abelian group. Since

$$\dim_G Z = \max(\dim_{G/\text{Tor } G} Z, \dim_{\text{Tor } G} Z)$$

for any metrizable space Z (see [Dy₁]), it suffices to prove

$$\dim_G(X \times Y) \leq \dim_G X + \dim_G Y + 1$$

first for torsion groups, and then for torsion-free groups.

Since \mathbf{Z}/\mathbf{p} is a ring with unity, the case $G = \mathbf{Z}/\mathbf{p}$ is taken care of. If $G = \mathbf{Z}/\mathbf{p}^\infty$ for some prime \mathbf{p} , then $G * G = G$, so $\dim_G(X \times Y) \leq \dim_G X + \dim_G Y + 1$ by Theorem 5.2. By [Dy₁],

$$\dim_G(X \times Y) \leq \dim_G X + \dim_G Y + 1$$

for any torsion group G .

Consider a torsion-free abelian group G . Let $\mathcal{L} = \{\mathbf{p} \mid \mathbf{p}G \neq G\}$. By [Dy₁],

$$\dim_{\mathbf{Z}_{\mathcal{L}}} Y \geq \dim_G Y \geq \dim_{\mathbf{Z}_{\mathcal{L}}} Y - 1$$

for any metrizable space X . Since $G \otimes \mathbf{Z}_{\mathcal{L}} = G$, by applying Theorem 5.2 to $H = \mathbf{Z}_{\mathcal{L}}$ one gets

$$\dim_G(X \times Y) \leq \dim_G X + \dim_{\mathbf{Z}_{\mathcal{L}}} Y \leq \dim_G X + \dim_G Y + 1.$$

□

Using Theorems 5.2-5.3, we can prove analogs of the Menger Theorem in extension theory (5.4-5.6):

Theorem 5.4. *Suppose X is a metrizable space, and Y is a σ -compact metrizable space. If $K \in AE(X)$ and $L \in AE(Y)$ are connected CW complexes, then*

$$SP^\infty(K \wedge L) \in AE(X \times Y).$$

Proof. We may assume Y is compact. Choose points $k \in K^0$, $l \in L^0$, and recall that the smash product $K \wedge L$ of K and L is $K \times L / K \vee L$, where $K \vee L = K \times l \cup k \times L$.

By Theorem 1.28 and Lemma 4 in [Ku] (p.34), we need to prove that

$$\dim_{H_q(K \wedge L; \mathbf{Z})}(X \times Y) \leq q$$

for all q . By the Kunneth formula ([Sp], p.235), there is an exact sequence

$$\begin{aligned} 0 \rightarrow [H(K, k; \mathbf{Z}) \otimes H(L, l; \mathbf{Z})]_q &\rightarrow H_q((K, k) \times (L, l); \mathbf{Z}) \\ &\rightarrow [H(K, k; \mathbf{Z}) * H(L, l; \mathbf{Z})]_{q-1} \rightarrow 0 \end{aligned}$$

for all q . Now, by Theorem 3.4 of [Dy₁],

$$\dim_{H_n(K, k; \mathbf{Z})} X \leq n$$

and

$$\dim_{H_n(L, l; \mathbf{Z})} Y \leq n$$

for all n . By Theorem 5.2,

$$\dim_{[H(K, k; \mathbf{Z}) \otimes H(L, l; \mathbf{Z})]_q}(X \times Y) \leq q$$

and

$$\dim_{[H(K, k; \mathbf{Z}) * H(L, l; \mathbf{Z})]_{q-1}}(X \times Y) \leq q$$

for all q . Finally (see Section 4 of [Ku] or [Dy₁]), $\dim_{H_q(K \wedge L; \mathbf{Z})}(X \times Y) \leq q$ for all q . □

Theorem 5.5. *Suppose X is a metrizable space of finite dimension, and Y is a σ -compact metrizable space of finite dimension. If $K \in AE(X)$ and $L \in AE(Y)$ are connected CW complexes, then*

$$K \wedge L \in AE(X \times Y).$$

Proof. Since both K and L are connected, we may assume that their 0-skeleta consist of single points k and l . Now, the smash product $K \wedge L$ has no 1-skeleton, and is simply connected. By Theorem 5.4 and Theorem 1.28, $K \wedge L \in AE(X \times Y)$. □

Theorem 5.6. *Suppose X and Y are compacta of finite dimension. If $K \in AE(X)$ and $L \in AE(Y)$ are CW complexes, then*

$$K \wedge L \in AE(X \times Y),$$

$$K \wedge L \in AE(X \wedge Y),$$

and

$$K * L \in AE(X * Y).$$

Proof. First, let us prove $K \wedge L \in AE(X \times Y)$. If both K and L are connected, this follows from Theorem 5.5. Assume K is not connected. In this case $S^0 \in AE(X)$, and any CW complex is an absolute extensor of X (see [Dra₄]). Without loss of generality we may assume that K is a simplicial complex with the weak topology. Since $Y \in S^k \wedge L$ and $Y \in I^k \wedge L$ for all $k \geq 0$ (see [Dra₅]), one gets $Y \in AE(K \wedge L)$ by [Ko] ($K \wedge L$ is a union of absolute extensors of Y with each finite intersection being an absolute extensor of Y). Suppose $f : A \rightarrow K \wedge L$ is continuous, and A is a closed subset of $X \times Y$. Given $x \in X$, one can extend f to $f_s : A \cup V_s \times Y \rightarrow K \wedge L$ for some open-closed neighborhood V_s of x (extend first f over $x \times Y$ and then over its neighborhood). Now, it is clear that f extends over $X \times Y$.

Since $X \wedge Y = X \times Y / X \vee Y$, we get $K \wedge L \in AE(X \wedge Y)$ and $\Sigma(K \wedge L) \in AE(\Sigma(X \wedge Y))$ (see [Dra₅], Proposition 6). Notice that $\Sigma(K \wedge L) \simeq K * L$. Since $K * L \in AE(X)$, $K * L \in AE(Y)$ and $K * L \in AE(X * Y - X \oplus Y)$, we get $K * L \in AE(X * Y)$ (see [Dra₅], Proposition 3). \square

Corollary 5.7. *Suppose X and Y are separable metrizable spaces of finite dimension. If $K \in AE(X)$ and $L \in AE(Y)$ are connected CW complexes, and L is homotopy dominated by a finite CW complex, then*

$$K \wedge L \in AE(X \times Y).$$

Proof. By Corollary 2.5 of [Dy₂] there is a compact space Y' containing Y such that $L \in AE(Y')$. By Theorem 5.5, $K \wedge L \in AE(X \times Y')$. Now, use Proposition 1.1 of [Dy₂]. \square

Theorem 5.8. *Suppose W is a separable metrizable space of finite dimension, Y is a σ -compact metrizable space of finite dimension, and G_1, \dots, G_m are countable abelian groups. Then, for any k so that $0 < k < \dim_{G_i} W - \dim_{G_i} Y$ for all $i \leq m$, there is a closed subset T of W such that*

$$\dim_{G_i} T = \dim_{G_i} W - k$$

and

$$\dim_{G_i}(T \times Y) = \dim_{G_i}(W \times Y) - k$$

for all $i \leq m$.

Proof. It suffices to consider the case $k = 1$. First, assume Y is compact. Let $\{V_i\}$ be a countable basis of Y . Suppose X is a metrizable space. Notice that $\dim_G(X \times Y) \leq n$ is equivalent to $H^{n+1-j}(X, X - U; H^j(Y, Y - V_i; G)) = 0$ for all i, j and all open subsets U of X , which is equivalent to $\dim_{H^j(Y, Y - V_i; G)} X \leq n - j$ for all i, j (see Section 5 of [Ku]). Given $j \leq \dim Y$, and given $n \leq m$, let $G(n, j)$ be the direct sum of all the groups $H^j(Y, Y - V_i; G_n)$, $i \geq 1$. Choose a closed subset T of W such that $\dim T = \dim_{\mathbf{Z}} T = \dim_{\mathbf{Z}} W - 1 = \dim W - 1$, $\dim_{G_i} T = \dim_{G_i} W - 1$ for $i \leq m$, and $\dim_{G(n, j)} T = \dim_{G(n, j)} W - 1$ if $\dim_{G(n, j)} W > 1$.

Clearly, $\dim_{G_i}(T \times Y) \leq \dim_{G_i}(W \times Y) - 1$ all $i \leq n$. Suppose $\dim_{G_s}(W \times Y) = d$ for some $s \leq n$. That means there is i, j such that $\dim_{H^j(Y, Y - V_i; G_s)} W = d - j$. Now, since $d - j > 1$ ($H^j(Y, Y - V_i; G_s) \neq 0$ means $j \leq \dim_{G_s} Y < \dim_{G_s} W - 1 \leq \dim_{G_s}(W \times Y) - 1 = d - 1$), then $\dim_{G(s,j)} T = d - j - 1$, and there is t such that $\dim_{H^j(Y, Y - V_i; G_s)} T = d - j - 1$, which implies $\dim_{G_s}(T \times Y) = d - 1$.

If Y is not compact, we express Y as the union of an increasing sequence Y_n of its compact subsets. Since $\dim_G Y = \max\{\dim_G Y_n \mid n \geq 1\}$ and $\dim_G(W \times Y) = \max\{\dim_G(W \times Y_n) \mid n \geq 1\}$ for all groups G , there is n such that $\dim_{G_i} Y = \dim_{G_i} Y_n$ and $\dim_{G_i}(W \times Y) = \dim_{G_i}(W \times Y_n)$ for $i \leq m$. Choose T for the space Y_n . \square

Corollary 5.9. *Suppose W is a separable metrizable space of finite dimension, and Y is a σ -compact metrizable space of finite dimension. Then, for any k , $0 < k < \dim W - \dim Y$, there is a closed subset T of W such that*

$$\dim T = \dim W - k$$

and

$$\dim(T \times Y) = \dim(W \times Y) - k.$$

Dranishnikov, Repovš and Ščepin [D-R-S₂] found the only example so far where the cartesian product $W \times Y$ of a 2-dimensional subset W of R^3 and a 1-dimensional continuum Y is 2-dimensional. We are uncertain if this pathology is restricted to low dimensions only.

Problem 5.10. *Suppose $n > 1$. Is there a separable metrizable space W of dimension $n + 1$ such that $\dim(W \times Y) = \dim W$ for some compactum Y of dimension n ?*

Notice that if $\dim Y = n$ and $\dim(W \times Y) = \dim W$ for some separable space of dimension greater than n , then Corollary 5.9 implies the existence of T closed in W such that $n + 1 = \dim T = \dim(T \times Y)$. Also, notice that if $\dim Y = \dim W > 0$, then $\dim(W \times Y) \geq \dim W + 1$ (see [Ku], Corollary on p.38).

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